Resummation of QCD corrections to the η_c decay rate

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We examine the ratio of the decay rate of the η_c into light hadrons to the decay rate into photons and find that most of the large next-to-leading-order (NLO) correction is associated with running of the strong coupling α_s. We resum such contributions by analyzing final-state chains of vacuum-polarization bubbles. We show that the nonperturbative parts of the bubble chains can be absorbed into a color-octet matrix element, once one has used contour deformations of the phase-space integrals to cancel certain contributions. We argue that these contributions are incompatible with the uncertainty principle. We also argue that perturbation theory is reliable only if one carries out the phase-space integrations before the perturbation summation. Our results are in good agreement with experiment and differ considerably from those that one obtains by applying the scale-setting method of Brodsky, Lepage, and Mackenzie to the NLO result.

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I. INTRODUCTION

In this paper we consider the decay of the heavy quarkonium η_c into light hadrons. According to the analysis of Ref. [1], the decay rate is given by the nonrelativistic quantum chromodynamics (NRQCD) factorization formula

$$\Gamma(\eta_c \to \mathrm{LH}) = \sum_n \frac{2\,\mathrm{Im}\, f_n}{m_c^{d_n-4}} \langle \eta_c | \mathcal{O}_n | \eta_c \rangle. \tag{1.1}$$

Here, m_c is the charm-quark mass, the $\mathcal{O}_n$ are operators in NRQCD with mass dimension d_n, and $2\,\mathrm{Im}\, f_n / m_c^{d_n-4}$ is a short-distance coefficient. A similar formula applies for the electromagnetic decay $\eta_c \to \gamma\gamma$.

In general, there are difficulties in obtaining accurate results from the factorization formula (1.1). The sum is an expansion in powers of the heavy-quark–antiquark relative velocity v. Only the first few operator matrix elements in that sum are known at all, and they are not known very accurately. The short-distance coefficients may, in principle, be computed in perturbation theory. However, the perturbation series is only asymptotic. In particular, it exhibits a factorial growth that is associated with the presence of renormalon singularities in its Borel transform [2]. Such singularities give rise to ambiguities in the evaluation of the perturbation series.

One can avoid some of these difficulties by considering the ratio of the decay rate for η_c into light hadrons to the decay rate for η_c into two photons. We denote this ratio by R. In R, the dependences of the decay rates on the operator matrix elements cancel in leading order in v, and the corrections of relative order v^2 also cancel [1]. These cancellations occur because, aside from an overall color factor, the tree-level Feynman diagrams for $Q\overline{Q} \to gg$ and $Q\overline{Q} \to \gamma\gamma$ are the same. One consequence of these cancellations is that renormalon ambiguities associated with initial-state virtual-gluon corrections also cancel [3,4] at the level of accuracy of relative order v^2. However, there is still the potential for renormalon ambiguities associated with final-state corrections to appear.

Through next-to-leading order in v^2, the ratio R is given by the ratio of the short-distance coefficients of the leading-order operators [1]. Through next-to-leading order (NLO) in α_s, one obtains [5,6]

$$R^{\mathrm{NLO}}(\mu) = R_0(\mu)\left[1 + \left(\frac{199}{6} - \frac{13\pi^2}{8} - \frac{8}{9}n_f\right)\frac{\alpha_s(\mu)}{\pi} + 2\alpha_s(\mu)\beta_0 \ln\frac{\mu^2}{4m_c^2}\right], \tag{1.2}$$

where

$$R_0(\mu) = \frac{81 C_F \alpha_s^2(\mu)}{32 N_c \alpha^2}, \tag{1.3}$$

m_c is the pole mass of the charm quark, n_f is the number of light quarks ($n_f = 3$ for charmonium), β_0 is the leading coefficient of the beta function, with $\beta_0 = (33 - 2n_f)/(12\pi)$, and, in QCD, $N_c = 3$, and $C_F = 4/3$.

Setting the factorization scale μ equal to $2m_c$ and taking $\alpha_s(2m_c) = 0.247$, one finds that

$$R^{\mathrm{NLO}}(2m_c) = 2.1 \times 10^3. \tag{1.4}$$

This is to be compared with the world-average experimental value [7]

$$R^{\mathrm{Expt}} = 3.3 \pm 1.3 \times 10^3. \tag{1.5}$$

However, the next-to-leading-order QCD correction is quite large. For $\alpha_s(2m_c) = 0.247$, it is about 1.1 times the leading-order term. Hence, there is some question as to the reliability of the perturbation expansion.

Furthermore, the dependence on the factorization scale is large. For example, taking $\mu = m_c$, with $\alpha_s(m_c) = 0.350$, we obtain

$$R^{\rm NLO}(m_c) = 5.0 \times 10^3, \tag{1.6}$$

which is a factor of 2.4 larger than $R^{\rm NLO}(2m_c)$.

One popular choice of scale is that given by the Brodsky-Lepage-Mackenzie (BLM) method [8]. In this approach, all terms proportional to n_f are absorbed into the running coupling $\alpha_s(\mu)$. For R, one obtains $\mu_{\rm BLM} \approx 0.52 m_c$ [8]. This yields a next-to-leading-order coefficient that is only about 0.5 times the leading-order coefficient. However,

$$R^{\rm NLO}(\mu_{\rm BLM}) = 9.9 \times 10^3, \tag{1.7}$$

which is much larger than either the theoretical result at $\mu = 2m_c$ or the experimental value.

We can gain some insight into the origins of the large next-to-leading-order correction and the strong scale dependence by re-writing Eq. (1.2) as follows:

$$R^{\rm NLO}(\mu) = R_0(\mu)\left\{1 + \left[\left(\frac{37}{2} - \frac{13\pi^2}{8}\right) + \pi\beta_0\left(\frac{16}{3} + 2\ln\frac{\mu^2}{4m_c^2}\right)\right]\frac{\alpha_s(\mu)}{\pi}\right\}. \tag{1.8}$$

In the next-to-leading-order correction, the first term in square brackets has a value of about 2.46, while the term $16\pi\beta_0/3$ has a value of 12. Hence, we see that most of the large next-to-leading-order correction is proportional to β_0. That is, it arises from the one-loop corrections associated with the running of α_s.

This suggests that we control the large next-to-leading-order correction by resumming, to all orders in perturbation theory, the contributions associated with the one-loop running of α_s. Such a resummation would have the added benefit of reducing the scale sensitivity, since the resummed contribution would be invariant under a change of scale, aside from two-loop corrections to the running of α_s. Unfortunately, there is some arbitrariness in the resummation procedure. The contributions to the running of α_s that arise from quark loops are contained entirely in chains consisting of quark vacuum-polarization bubbles connected by gluon propagators. However, the contributions that arise from gluon loops may, depending on the gauge, be associated with diagrams other the gluon vacuum polarization.

One unambiguous procedure for carrying out such a resummation is the method of ''naive non-Abelianization'' (NNA) [9]. In this method, one resums the chains of quark-loop vacuum-polarization bubbles and then accounts for the gluonic contributions by promoting the quark-loop contribution to β_0 to the full QCD β_0. This method is a generalization of the BLM scale-setting method. However, as we shall see, it leads to a very different numerical result. An alternative method for carrying out the resummation is to resum both the quark-loop and gluon-loop vacuum-polarization bubble chains in the background-field gauge [10]. In this gauge, all of the corrections associated with the running of α_s are contained in the quark and gluon vacuum-polarization diagrams. It turns out that the background-field-gauge method gives numerical results that are close to those of the NNA method.

Vacuum-polarization insertions first appear, in relative order α_s, in the final-state gluon legs. Therefore, in order to resum the leading bubble-chain corrections, we need consider only vacuum-polarization bubble chains in the two final-state gluon legs. By ''leading bubble-chain corrections,'' we mean those corrections that contain one power of the large coefficient in the second term in square brackets in Eq. (1.8) for each power of α_s.

The integrals over the final-state phase space include regions in which the constituents of the vacuum-polarization diagrams (quarks or gluons) have a small invariant mass. Hence, the calculation is, in principle, sensitive to nonperturbative contributions. In the bubble-chain series, this sensitivity may manifest itself as a factorial growth that is characteristic of the presence of renormalon singularities in the Borel transform of the series.

In the case of a single chain of vacuum-polarization bubbles, we show, by making use of a contour-deformation argument in the complex invariant-mass plane, that the low-virtuality contributions, and, hence, the factorial growth, cancel when one carries out the phase-space integration before carrying out the perturbation summation. Somewhat surprisingly, even when factorial growth in the series cancels, there is an ambiguity that arises in the bubble-chain series: one obtains different results, depending on whether one carries out the perturbation summation before the phase-space integration, or vice versa. We argue that the perturbation series is reliable only when one carries out the phase-space integration first.

For the more general case of two bubble chains, there is a true sensitivity to the low-virtuality region, even after contour-deformation arguments have been applied. Indeed, the perturbation series exhibits a factorial growth that is characteristic of the presence of renormalon singularities in its Borel transform. We show that the low-virtuality contributions can be identified with contributions to the decay rate that arise from the matrix element of a color-octet NRQCD operator. Hence, by taking into account the color-octet contribution, one can systematically remove the low-virtuality region from the perturbative calculation. In this analysis, contour deformation arguments are crucial to demonstrate the cancellation of certain low-virtuality contributions that do not correspond to matrix elements of NRQCD operators. We argue that such contributions are inconsistent with the uncertainty principle.

The color-octet matrix element is inherently nonperturbative in nature. We do not calculate its contribution, which is of relative order v^3. Instead, we estimate the size of the contribution and treat it as an uncertainty in the calculation.

The remainder of this paper is organized as follows. In Sec. II, we compute the contributions of the vacuum-polarization bubble chains to all orders in α_s. We re-arrange the real and virtual contributions to obtain manifestly infrared finite expressions. In Sec. III, we discuss the contributions from the nonperturbative regions of small invariant mass, first for a single bubble chain and then for two bubble

chains. We identify the nonperturbative contributions that remain after the contour-deformation cancellation with contributions to the matrix elements of a color-octet operator. Section IV contains a re-arrangement of the bubble-chain contributions that is efficient for numerical evaluation. In Sec. V, we present our numerical results, and in Sec. VI we give a summary and a discussion of our findings.

II. THE BUBBLE-CHAIN CONTRIBUTIONS TO *R*

In this section, we compute the contributions to R from the insertion of any number of vacuum-polarization bubbles into the two final-state gluons in the lowest-order Feynman diagrams for $Q\overline{Q}\to gg$.

These contributions are obtained by cutting the diagrams for the heavy-quark–antiquark forward scattering amplitude through light-quark and gluon final states in all possible ways. A cut can pass through two vacuum-polarization bubbles, a final-state gluon and a vacuum-polarization bubble, or two final-state gluons. We call such contributions real-real, virtual-real, and virtual-virtual, respectively. These contributions are separately infrared divergent, but the Kinoshita-Lee-Nauenberg (KLN) theorem [11] ensures that the infrared divergences cancel in the complete expression for the decay width.

A. The effects of bubble chains in the hadronic cross section

In NRQCD, at leading order in v, the decay of a $Q\overline{Q}\ {}^1S_0$ state proceeds through the color-singlet operator

$$\mathcal{O}_1({}^1S_0)=\psi^\dagger\chi\chi^\dagger\psi, \tag{2.1}$$

where ψ is the two-component Pauli spinor that annihilates the charm quark and $\chi^\dagger$ is the two-component Pauli spinor that annihilates the charm antiquark. According to the NRQCD factorization formula [1], the contribution of this operator to the decay rate in NRQCD is given by

$$\Gamma(H)=2\,\mathrm{Im}\left[\frac{f_1({}^1S_0)}{m_c^2}\right]\langle H|\mathcal{O}_1({}^1S_0)|H\rangle, \tag{2.2}$$

where H is a hadronic state.

We can compute the short-distance coefficient $2\,\mathrm{Im}[f_1({}^1S_0)/m_c^2]$ in perturbation theory by taking the state H to be a free $Q\overline{Q}$ state. Then, Eq. (2.2) becomes $\Gamma(Q\overline{Q})$, the rate for the annihilation of a free heavy-quark–antiquark pair into gluons through the operator $\mathcal{O}_1({}^1S_0)$.

At leading order in the heavy-quark–antiquark velocity v in the quark-antiquark center-of-mass frame, the amplitude for the decay of a heavy quark and antiquark in a color-singlet ${}^1S_0$ state with momenta p_1 and p_2, respectively, into two massive gluons with momenta k and l, polarizations μ and ν, and color indices a and b, respectively, is given by

$$A_{ab}^{\mu\nu}(Q\overline{Q})=\frac{1}{\sqrt{N_c}}\,\mathrm{Tr}(T_aT_b)2\sqrt{2}g^2\epsilon^{\nu\rho\mu0}k_\rho\,\frac{1}{2m_ck_0+k^2}, \tag{2.3}$$

where g is the strong coupling constant, T_a is an $SU(3)$ color matrix in the fundamental representation, and the trace is over the indices of the $SU(3)$ matrices. In deriving Eq. (2.3), we have made use of the spin-0 projector, accurate to leading order in v:

$$\sum_{s_{1z},s_{2z}} u(p_1,s_1)\overline{v}(-p_2,s_2)\langle s=0|s_1s_2\rangle = \frac{1}{4\sqrt{2}m_c^2}(\not{p}_1+m_c)\frac{1+\gamma_0}{2}\,\gamma_5(\not{p}_2-m_c). \tag{2.4}$$

Taking the absolute square of the quantity in Eq. (2.3), inserting vacuum-polarization bubbles in the gluon propagators, dividing by two for Bose statistics, and integrating over the phase space, we obtain the bubble-chain contribution to the $Q\overline{Q}$ decay width:

$$\Gamma^{\mathrm{Bub}}(Q\overline{Q})=2g^4C_F\sum_{m,n=0}^{\infty}\int\frac{d^4k}{(2\pi)^4}\,\theta(k_0)\int\frac{d^4l}{(2\pi)^4}\,\theta(l_0)\epsilon^{\nu\rho\mu0}k_\rho\epsilon_{\nu\sigma\mu0}k^\sigma\left(\frac{1}{2mk_0+k^2}\right)^2 2\,\mathrm{Im}[K^{(m)}(x)]2\,\mathrm{Im}[K^{(n)}(y)] \times(2\pi)^4\delta^4(p_1+p_2-k-l), \tag{2.5}$$

where

$$x=k^2/(4m_c^2), \tag{2.6a}$$

$$y=l^2/(4m_c^2). \tag{2.6b}$$

Aside from a polarization-tensor factor,

$$iK^{(m)}(x)=[-i\Pi(x)]^m\frac{i}{4m_c^2(x+i\epsilon)} \tag{2.7a}$$

is the amplitude for the mth-order contribution to the bubble-chain gluon propagator. We also define the complete propagator

$$K(x)=\sum_{m=0}^{\infty}K^{(m)}(x). \tag{2.7b}$$

$\Pi(x)$ is defined by

$$\Pi_{\mu\nu}(k)=(k^2g_{\mu\nu}-k_\mu k_\nu)\Pi(x), \tag{2.8}$$

where $\Pi_{\mu\nu}(k)$ is the amplitude for a vacuum-polarization bubble with external momentum k and polarization indices μ and ν. We note the useful kinematic relations

$$k_0 = m_c(1+x-y), \tag{2.9a}$$

$$l_0 = m_c(1-x+y), \tag{2.9b}$$

$$\mathbf{k}^2 = \mathbf{l}^2 = m_c^2(1+x^2+y^2-2x-2y-2xy). \tag{2.9c}$$

Using Eq. (2.5) and

$$\langle Q\overline{Q}|\mathcal{O}_1(^1S_0)|Q\overline{Q}\rangle = 2N_c, \tag{2.10}$$

we compare the left and right sides of Eq. (2.2) to obtain the short-distance coefficient

$$2\,\mathrm{Im}\left[\frac{f_1(^1S_0)}{m_c^2}\right] = \frac{1}{2N_c}\Gamma^{\mathrm{Bub}}(Q\overline{Q}). \tag{2.11}$$

Then, taking the hadronic state H in Eq. (2.2) to be the η_c, we obtain the decay width of the η_c into two gluons in the bubble-chain approximation:

$$\Gamma^{\mathrm{Bub}}(\eta_c) = \frac{g^4 C_F}{N_c}\sum_{m,n=0}^{\infty}\int\frac{d^4k}{(2\pi)^4}\theta(k_0)\int\frac{d^4l}{(2\pi)^4}\theta(l_0)\epsilon^{\nu\rho\mu 0}k_\rho\epsilon_{\nu\sigma\mu 0}k^\sigma\left(\frac{1}{2mk_0+k^2}\right)^2 2\,\mathrm{Im}[K^{(m)}(x)]2\,\mathrm{Im}[K^{(n)}(y)]$$
$$\times(2\pi)^4\delta^4(p_1+p_2-k-l)\langle\eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle. \tag{2.12}$$

In the η_c width for decay into two gluons (2.12), we change variables by making use of Eq. (2.6), which implies that $2k_0dk_0\theta(k_0) = 4m_c^2dx\,\theta(x)$ and $2l_0dl_0\theta(l_0) = 4m_c^2dy\,\theta(y)$, and Eq. (2.9). We use the energy-momentum-conserving δ functions to carry out the integrations over the 3-momenta. Then, dividing by the decay width into two photons,

$$\Gamma_{\mathrm{EM}}(\eta_c) = \frac{32\pi\alpha^2}{81m_c^2}\langle\eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle, \tag{2.13}$$

we obtain the contribution to R:

$$R^{\mathrm{Bub}} = R_0\sum_{m,n=0}^{\infty}\int_0^1\frac{dx}{2\pi}\int_0^1\frac{dy}{2\pi}\,2\,\mathrm{Im}[4m_c^2K^{(m)}(x)]$$
$$\times 2\,\mathrm{Im}[4m_c^2K^{(n)}(y)]f(x,y)\theta(1-\sqrt{x}-\sqrt{y}), \tag{2.14}$$

where

$$f(x,y) = \frac{[1-2(x+y)+(x-y)^2]^{3/2}}{(1-x-y)^2}. \tag{2.15}$$

In order to compute R^{Bub} from Eq. (2.14), we need the value of a single vacuum-polarization bubble. A standard calculation of the light-quark bubble in dimensional regularization yields

$$\Pi(x) = \frac{i}{\epsilon}\alpha_s\beta_0(bx^{-\epsilon}-a), \tag{2.16}$$

where

$$b = \left[1+\epsilon\left(-\gamma+\ln 4\pi+\frac{5}{3}+\ln\frac{\mu^2}{4m_c^2}\right)+O(\epsilon^2)\right]e^{i\pi\epsilon}, \tag{2.17a}$$

$$a = 1+\epsilon\left(-\gamma+\ln 4\pi+\frac{5}{3}+C\right)+O(\epsilon^2). \tag{2.17b}$$

In Eq. (2.16), we have renormalized the ultraviolet divergence by subtracting the ultraviolet pole in ϵ and some associated constants. ϵ now plays the role of an infrared regulator ($\epsilon<0$). C is a subtraction-scheme-dependent constant; in the modified minimal subtraction scheme ($\overline{\mathrm{MS}}$), $C=-5/3$. We note that the phase of b in Eq. (2.17a) and the reality of a are exact to all orders in ϵ. For the light-quark bubble, $\beta_0 = -2n_f/(12\pi)$. However, in employing NNA, we promote β_0 to the QCD value $\beta_0 = (33-2n_f)/(12\pi)$.

An alternative to the NNA procedure is to compute the vacuum polarization, including both the quark and gluon contributions, in the background-field gauge. The motivation for this approach is that, in the background-field gauge, the logarithmic dependence on the renormalization scale μ is contained entirely in the vacuum-polarization diagrams [12]. In the background-field gauge, the vacuum-polarization contribution still takes the form (2.16). The coefficients a and b are computed in Appendix A and are given in Eq. (A7).

B. Decomposition into real and virtual contributions

It is convenient to separate R^{Bub} into real-real, real-virtual, and virtual-virtual contributions. First, we make use of the identity

$$\mathrm{Im}(AB) = (\mathrm{Im}\,A)B^* + A(\mathrm{Im}\,B) \tag{2.18}$$

to identify the real and virtual parts of a single propagator:

$$\mathrm{Im}[4m_c^2K^{(m)}(x)]=\frac{1}{x-i\epsilon}\mathrm{Im}[-i\Pi(x)]^m - \pi\delta(x)[-i\Pi(0)]^m. \quad (2.19)$$

Then, we substitute this result into Eq. (2.14) to obtain

$$R^{\mathrm{Bub}}=R_0(G_{2R}+2G_VG_{1R}+G_V^2), \quad (2.20)$$

where the three terms in parentheses correspond, respectively, to the real-real, real-virtual, and virtual-virtual contributions.

$$G_{2R}=\sum_{m,n=1}^{\infty}\int_0^1\frac{dx}{2\pi x}\int_0^1\frac{dy}{2\pi y}f(x,y)I_R^{(m)}(x)I_R^{(n)}(y) \times\theta(1-\sqrt{x}-\sqrt{y}) \quad (2.21)$$

is the contribution to the amplitude from two chains of real vacuum-polarization bubbles.

$$I_R^{(m)}(x)=-2\,\mathrm{Im}[-i\Pi(x)]^m \quad (2.22)$$

is the mth-order contribution to the amplitude for a single real bubble chain with virtuality x.

$$G_{1R}=\sum_{m=1}^{\infty}\int_0^1\frac{dx}{2\pi x}f(x,0)I_R^{(m)}(x) \quad (2.23)$$

is the amplitude for a single real bubble chain, integrated over its virtuality and weighted with the heavy-quark and phase-space factor, with the virtuality of the other chain set to zero.

$$G_V=\sum_{m=0}^{\infty}[-i\Pi(0)]^m \quad (2.24)$$

is the amplitude for a single chain of virtual vacuum-polarization insertions. In the last term of Eq. (2.20), we have used the fact that $f(0,0)=1$.

C. Manifestly infrared-finite expressions

In computing the expression for R^{Bub} in Eq. (2.20), it is useful to re-arrange the individual contributions to make the expression manifestly infrared finite. First, we define the quantity

$$G_1=G_{1R}+G_V. \quad (2.25)$$

Since G_{1R} and G_V are the real and virtual contributions to a single vacuum-polarization-bubble chain, the KLN theorem guarantees that G_1 is infrared finite. We verify the finiteness of G_1 by explicit calculation in Appendix B. It is also useful to define

$$G_2=G_{2R}-G_{1R}^2 =\sum_{m,n=1}^{\infty}\left[\int_0^1\frac{dx}{2\pi x}\int_0^1\frac{dy}{2\pi y}\,f(x,y)I_R^{(m)}(x)I_R^{(n)}(y) \times\theta(1-\sqrt{x}-\sqrt{y})-\int_0^1\frac{dx}{2\pi x}\,f(x,0)I_R^{(m)}(x) \times\int_0^1\frac{dy}{2\pi y}\,f(0,y)I_R^{(n)}(y)\right]. \quad (2.26)$$

Since the contributions from the regions of integration near $x=0$ and/or $y=0$ are equal in the first and second terms in Eq. (2.26), G_2 is infrared finite. Then, using Eqs. (2.20), (2.23) and (2.26), we can write

$$R^{\mathrm{Bub}}=R_0(G_1^2+G_2). \quad (2.27)$$

Each term in Eq. (2.27) is infrared finite.

The quantity G_1 is computed in Appendix B. The result is

$$G_1=\frac{1}{\pi\alpha_s\beta_0}\arctan\frac{\pi\alpha_s\beta_0}{1-\alpha_s\beta_0 d}+\sum_{n=1}^{\infty}\int_0^1 dx\,g_n(x), \quad (2.28)$$

where

$$g_n(x)=\lim_{\epsilon\to0}\frac{1}{\pi}\,\mathrm{Im}[-i\Pi(x)]^n=\frac{1}{\pi}\,\mathrm{Im}[\alpha_s\beta_0(d-\ln x +i\pi)]^n, \quad (2.29)$$

and

$$d=\lim_{\epsilon\to0}\mathrm{Re}(b-a)/\epsilon. \quad (2.30)$$

In naive non-Abelianization,

$$d=\ln[\mu^2/(4m_c^2)]-C, \quad (2.31)$$

while in the background-field gauge

$$d=\frac{1}{\beta_0\pi}\left[\frac{67}{12}-\frac{5}{18}n_f-\frac{3}{4}(\xi^2-1)-\frac{5}{3}-C\right]+\ln\frac{\mu^2}{4m_c^2}. \quad (2.32)$$

By re-arranging Eq. (2.26), we can obtain an expression for G_2 in which every integral is separately infrared finite, and, hence, we can take ϵ to zero:

$$G_2=\sum_{m,n=1}^{\infty}\left\{-\int_0^1\frac{dx}{x}(1-x)g_m(x)\int_0^1\frac{dy}{y}(1-y)g_n(y)\right.$$
$$\times\theta(\sqrt{x}+\sqrt{y}-1)+\int_0^1\frac{dx}{x}g_m(x)$$
$$\times\int_0^1\frac{dy}{y}g_n(y)[f(x,y)-(1-x)(1-y)]$$
$$\left.\times\theta(1-\sqrt{x}-\sqrt{y})\right\}. \quad (2.33)$$

Equations (2.27), (2.28), and (2.33) are the basis for the calculations in this paper.

Taking into account the μ dependence of d [Eq. (2.30)], one can verify straightforwardly that $\alpha_s(\mu)G_1(\mu)$ and $\alpha_s^2(\mu)G_2(\mu)$ are μ independent if the running coupling constant $\alpha_s(\mu)$ satisfies the one-loop renormalization-group equation

$$(\mu\partial/\partial\mu)\alpha_s(\mu)=2\beta_0\alpha_s(\mu). \quad (2.34)$$

III. THE NONPERTURBATIVE REGIONS

So far, we have treated the effects of vacuum-polarization insertions as if they were entirely perturbative in nature. However, we expect perturbation theory to break down in the regions in which the gluon virtualities x and y are near zero, and, for on-shell gluons, in the regions in which energies l_0 and k_0 are near zero.

A. Single bubble chain

We begin by investigating the region of small virtuality for a single bubble chain. That is, we consider the region of small x in the second term of G_1 in Eq. (2.28), namely,

$$G_{1b}=\sum_{n=1}^{\infty}\int_0^1 dx\,g_n(x)$$
$$=\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^1 dx\,\mathrm{Im}[\alpha_s\beta_0(d-\ln x+i\pi)]^n. \quad (3.1)$$

Near $x=0$, the $\ln x$ in G_{1b} becomes unbounded, and the series fails to converge. This means that, in principle, the result that one obtains by carrying out the integration before the summation may be different from the result that one obtains with the opposite order of operations. In fact, as we shall see, interchange of these operations does produce a finite difference in the result.

Since

$$\int_0^{\lambda}dx\ln^n x\sim n!, \quad (3.2)$$

one might expect the terms in the series in Eq. (3.1) to exhibit factorial growth, arising from the region of integration near $x=0$. In fact, a careful calculation shows that the factorial growth cancels when one takes the imaginary part. Factorial growth in a bubble-chain series is associated with a renormalon singularity in the Borel transform of the series. Explicit calculation of the Borel transform of G_{1b} shows that the renormalon singularity in the Borel plane vanishes when one takes the imaginary part [2].

It is useful, in understanding both the dependence on the order of operations and the absence of factorial growth, to re-write the expression for G_{1b} as a contour integral:

$$G_{1b}=\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_C dz\{\alpha_s\beta_0[d-\ln(-z)]\}^n. \quad (3.3)$$

The contour C runs from $z=1-i\epsilon$ to $z=0$ to $z=1+i\epsilon$. Term by term in the series, we can, without encountering any singularities, deform the contour C into the contour C', which is a circle of unit radius, centered at the origin, and traversed in the clockwise direction from $z=1-i\epsilon$ to $z=1+i\epsilon$. Along the new contour C', the magnitude of z is never small. Hence, the logarithm in Eq. (3.3) is bounded, and it is clear that there is no factorial growth in the series. Furthermore, the perturbation sum is now uniformly convergent, and we can interchange the summation and integration without affecting the result. Hence,

$$G_{1b}=\frac{1}{2\pi i}\int_{C'}dz\,\frac{1}{1-\alpha_s\beta_0[d-\ln(-z)]}. \quad (3.4)$$

Suppose, on the other hand, that we carry out the perturbation summation before deforming the contour. Then we have

$$G'_{1b}=\frac{1}{\pi}\int_0^1 dx\,\mathrm{Im}\left[\frac{1}{1-\alpha_s\beta_0(d-\ln x+i\pi)}\right]$$
$$=\frac{1}{2\pi i}\int_C dz\,\frac{1}{1-\alpha_s\beta_0[d-\ln(-z)]}. \quad (3.5)$$

If we deform the contour to C', then we encounter the Landau pole at

$$z_0=-\exp\left[d-\frac{1}{\alpha_s\beta_0}\right], \quad (3.6)$$

and we must include its residue in the result:

$$G'_{1b}=\frac{1}{2\pi i}\int_{C'}dz\,\frac{1}{1-\alpha_s\beta_0[d-\ln(-z)]}+\frac{z_0}{\alpha_s\beta_0}. \quad (3.7)$$

Then, using Eqs. (3.4) and (3.7), we have

$$G_{1b} = G'_{1b} - \frac{z_0}{\alpha_s \beta_0}. \tag{3.8}$$

The contribution from the residue at the Landau pole [the second term in Eq. (3.8)] is just the difference in results between carrying out the integration before the summation and carrying out the summation before the integration. It displays an essential singularity in α_s and has the characteristic form of a power correction associated with a renormalon singularity. However, as we have seen, in this case, there is no renormalon singularity associated with the Borel transform of the series.

The ambiguity in the result for G_{1b} raises a question as to which order of operations, if either, is correct. If we carry out the integration before the summation, then the contour-deformation argument shows that there is no contribution from the region of small virtuality. In this case, asymptotic freedom suggests that the perturbation series is reliable. On the other hand, if we carry out the summation first, then there is no reason to trust the resulting expression: for fixed x sufficiently small, we are outside the radius of convergence of the geometric series and outside the regime of asymptotic freedom. Indeed, the Landau pole in the summed series is most likely an artifact of perturbation theory that would not persist in the true solution of QCD and, hence, the low-virtuality contribution from the residue at the pole is spurious.[1] Therefore, we conclude that the correct order of operations is to perform the integration first and then the summation.

We note that the procedure of integrating over the gluon virtualities before carrying out the perturbation summation is equivalent to the standard Borel-transform method [2] for evaluating the contribution of a bubble chain. In that method, one takes the Borel transform of the series, sums the transformed series, carries out the integration over the phase space, and, finally, takes the inverse Borel transform. Since the initial Borel transform renders the series convergent for all virtualities and absolutely integrable, one can interchange the order of operations, carrying out the integration first and then summing the Borel-transformed series. Furthermore, term by term in the series, the integration and Borel-transformation commute. Therefore, the standard Borel-transform method is equivalent to carrying out the integration, taking the Borel transform, summing the series, and then taking the inverse transform. This, by the definition of the inverse transform, is equivalent to carrying out the integration and then summing the perturbation series.

In Appendix B, we have computed the first term of G_1 in Eq. (2.28) by performing the integration over x before the summation of the perturbation series. In this case, that is the only sensible order of operations: the perturbation series is not convergent for small values of the dimensional regulator ϵ until one has canceled the infrared divergences by carrying out the integration over x in the real contribution and adding the real and virtual contributions order by order.

We must also consider the possibility that, when x (y) is sufficiently close to unity, the energy l_0 (k_0) of the on-shell gluon may be too small to lie within the perturbative regime. [Recall that, although we have written Eq. (3.1), for compactness, only in terms of x, it derives from expressions that are symmetric in x and y.] Let us, for the moment, set aside the contributions from the nonperturbative regions of small l_0 (k_0) by imposing a restriction on the region of integration:

$$x, y < (1 - \sqrt{\delta})^2. \tag{3.9}$$

Here, δ is a positive parameter that we take to be much less than unity, but not so small that $4m_c^2\delta$ is outside the perturbative regime. Using Eq. (2.9), we see that the restriction (3.9) implies that

$$l_0^2 \geqslant m_c^2 (2\sqrt{\delta} - \delta)^2 \approx 4 m_c^2 \delta, \tag{3.10a}$$

$$k_0^2 \geqslant m_c^2 (2\sqrt{\delta} - \delta)^2 \approx 4 m_c^2 \delta. \tag{3.10b}$$

As we shall see in Sec. III C, the nonperturbative regions that we have set aside in imposing this restriction yield contributions of the form of one-loop corrections to the NRQCD matrix element of a color-octet 4-fermion operator.

B. Two bubble chains

The quantity G_2 [Eq. (2.33)] arises the from real-real contribution of the bubble chains. Hence, both virtualities x and y are nonzero. Furthermore, there are constraints on x and y. The first term in braces is subject to the constraint $\sqrt{x} + \sqrt{y} \geqslant 1$; the second term in braces is subject to the constraint $\sqrt{x} + \sqrt{y} \leqslant 1$. Owing to these constraints, one cannot immediately apply the contour-deformation argument of the last subsection to avoid the regions of small virtuality. For the first term in braces, when x is near unity, y ranges from unity almost to zero $[(1 - \sqrt{x})^2]$. One could write the integral over y as a contour integral in the complex plane. However, the lower end points are tied down at $y = (1 - \sqrt{x})^2 \pm i\epsilon$, and, hence, one cannot deform the contour out of the small-y region. For the second term in braces, when x is near unity, then y ranges from zero up to a small number $[(1 - \sqrt{x})^2]$. Again, one could write the integral over y as a contour integral in the complex plane, but owing to the fixed end points at $y = (1 - \sqrt{x})^2 \pm i\epsilon$, one cannot deform the contour out of the small-y region. We deal with this difficulty by setting aside, temporarily, parts of the region of integration by imposing, again, the restriction (3.9).

[1]Note that the situation is less clear in the present case than, for example, in the analysis of the τ hadronic width, for which similar contour deformation arguments have been made [13]. The τ width possesses a spectral (Källen-Lehmann) representation, which gives one some information about its analytic properties. The gluon propagator, being a colored object, possesses no spectral representation.

For the first term in braces in Eq. (2.33), the restriction (3.9), combined with the constraint $\sqrt{x}+\sqrt{y}\geq 1$, guarantees that

$$x,y \geq \delta. \tag{3.11}$$

Hence, the x and y integrations lie entirely within the perturbative regime.

For the second term in braces in Eq. (2.33), we may use the restriction (3.9) to decompose the region of integration as follows:

$$\begin{aligned}
&\int_0^{(1-\sqrt{\delta})^2} dx \int_0^{(1-\sqrt{\delta})^2} dy\ \theta(1-\sqrt{x}-\sqrt{y}) \\
&\quad = \int_\delta^{(1-\sqrt{\delta})^2} dx \int_\delta^{(1-\sqrt{\delta})^2} dy\ \theta(1-\sqrt{x}-\sqrt{y}) \\
&\qquad + \int_\delta^{(1-\sqrt{\delta})^2} dx \int_0^{\delta} dy + \int_0^{\delta} dx \int_\delta^{(1-\sqrt{\delta})^2} dy \\
&\qquad + \int_0^{\delta} dx \int_0^{\delta} dy.
\end{aligned} \tag{3.12}$$

In the first term on the right side of Eq. (3.12), both integrations lie entirely within the perturbative regime. In the second term on the right side of Eq. (3.12), the x integration lies entirely within the perturbative regime, while the y integration does not. However, we can write the y integral, as in the preceding subsection, as an integral over a contour running from $\delta - i\epsilon$ to zero to $\delta + i\epsilon$. We can then deform this contour into a circle of radius δ centered at the origin that is traversed in the clockwise direction from $\delta - i\epsilon$ to $\delta + i\epsilon$. It is easy to see that one encounters no singularities in the integrand in performing this contour deformation. Hence, we conclude that there are no contributions to the second term on the right side of Eq. (3.12) from the region of small y. Similarly, for the third term on the right side of Eq. (3.12), we can write that the x integral as a contour integral and deform the contour out of the region of small x. Finally, for the fourth term on the right side of Eq. (3.12), we can write both the x and y integrals as contour integrals and deform both contours out of the region of small virtuality. Therefore, we conclude that, once we have imposed the restriction (3.9), the contributions to the second term in braces in Eq. (2.33) lie entirely within the perturbative regime.

Again, we shall find in Sec. III C that the contributions that we have set aside by virtue of the restriction (3.9) correspond to one-loop corrections to the NRQCD matrix element of a color-octet 4-fermion operator. We note that the contributions to G_2 that we have set aside, when treated in perturbation theory, yield renormalon singularities. Such singularities are a signal that the contribution cannot be computed reliably in perturbation theory. In Appendix C, we compute the leading renormalon singularity that arises from these contributions to G_2.

C. Contribution of the color-octet NRQCD matrix element

Now let us discuss the nonperturbative contributions that we have set aside by virtue of the restriction on the region of integration (3.9). For the contributions that have been set aside, we have either $x \geq (1-\sqrt{\delta})^2$ or $y \geq (1-\sqrt{\delta})^2$, but not both, since this would violate the kinematic limit $\sqrt{x}+\sqrt{y} < 1$. Hence, one of the final-state bubble chains is highly virtual compared to the momentum of the other bubble chain or the relative momentum of the incoming heavy $Q\overline{Q}$ pair. Therefore, we may approximate these contributions by shrinking the highly virtual bubble chain to a point, replacing the bubble chain and its $Q\overline{Q}$ interaction vertices with a four-fermion interaction. Corrections to this approximation may be taken into account by including four-fermion interactions containing derivatives with respect to the heavy-quark separations. It follows that the contributions that have been set aside are actually one-bubble-chain corrections to the matrix element of a four-fermion operator in the η_c state.

In NRQCD at leading order in v, the four-fermion operator that arises when one shrinks a bubble chain to a point is the color-octet, spin-triplet operator

$$\mathcal{O}_8(^3S_1) = \psi^+ \sigma^i T^a \chi \chi^\dagger \sigma^i T^a \psi. \tag{3.13}$$

We now verify that the one-bubble-chain corrections to the contributions of this operator to R do indeed reproduce, at leading order in v, the contributions that we have set aside.

According to the NRQCD factorization formula [1] the contribution of this operator to the decay rate factors:

$$\Gamma_8^{\rm Bub}(\eta_c) = 2\ {\rm Im}\left[\frac{f_8(^3S_1)}{m_c^2}\right] \langle \eta_c | \mathcal{O}_8(^3S_1) | \eta_c \rangle, \tag{3.14}$$

where ${\rm Im}[f_8(^3S_1)/m_c^2]$ is the short-distance coefficient. By matching Eq. (3.14) to the imaginary part of the amplitude in full QCD for $Q\overline{Q} \to Q\overline{Q}$ via a single bubble chain, one finds, in the bubble-chain approximation, that

$$2\ {\rm Im}\left[\frac{f_8(^3S_1)}{m_c^2}\right] = -2g^2\ {\rm Im}[K(1)], \tag{3.15}$$

where $K(x)$ is defined in Eq. (2.7).

The color-octet matrix element is fully determined by the NRQCD effective Lagrangian. It is nonperturbative in nature and must be determined phenomenologically or through methods such as lattice QCD. However, our aim here is to compute the perturbative contribution to this matrix element in the bubble-chain approximation so that we can compare it with the part of our perturbative calculation of $R^{\rm Bub}$ that we have set aside.

In NRQCD, the one-bubble-chain correction to the operator matrix element arises, in leading order in the heavy-quark velocity v, through the insertion of $\sigma \cdot B$ operators into any two of the four quark lines. In the Feynman gauge, this contribution can be written as

$$\langle H|\mathcal{O}_8(^3S_1)|H\rangle = 4g^2\frac{C_F}{2N_c}\sum_{m=0}^{\infty}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}iK^{(m)}(x)\frac{(\boldsymbol{\sigma}\times\mathbf{k})_i}{2m_c}\frac{(-\boldsymbol{\sigma}\times\mathbf{k})_i}{2m_c}\left(\frac{i}{-k_0-\mathbf{k}^2/(2m_c)+i\epsilon}\right)^2\langle H|\mathcal{O}_1(^1S_0)|H\rangle$$
$$= -4g^2\frac{C_F}{2N_c}\sum_{m=0}^{\infty}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}iK^{(m)}(x)\left(\frac{1}{-k_0-\mathbf{k}^2/(2m_c)+i\epsilon}\right)^2\frac{\mathbf{k}^2}{2m_c^2}\langle H|\mathcal{O}_1(^1S_0)|H\rangle. \tag{3.16}$$

The superscript Λ is a reminder that the NRQCD effective theory requires an ultraviolet cutoff. H denotes a hadronic state.

The k_0 integral can be re-written by making use of the techniques of contour integration. Closing the k_0 contour in the lower half plane, we pick up the residues at the poles and the discontinuities across the cuts in $K^{(m)}(x)$. The contribution of these residues and discontinuities can be written in terms of the imaginary part of $K^{(m)}(x)$. Then we have

$$\langle H|\mathcal{O}_8(^3S_1)|H\rangle = \frac{g^2C_F}{N_cm_c^2}\sum_{m=0}^{\infty}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\theta(k_0)\times[-2\,\mathrm{Im}\,K^{(m)}(x)]\times\left(\frac{1}{-k_0-\mathbf{k}^2/(2m_c)+i\epsilon}\right)^2\mathbf{k}^2\times\langle H|\mathcal{O}_1(^1S_0)|H\rangle. \tag{3.17}$$

Using Eq. (2.9), we can replace integration variables k_0 and $\mathbf{k}$ with x and y. We note that

$$dk_0d|\mathbf{k}|^2 = 4m_c^3dx\,dy. \tag{3.18}$$

Making this change of variables in Eq. (3.17) and multiplying by the short-distance quantity in Eq. (3.15), we obtain the one-bubble-chain correction to the rate:

$$\Gamma_8^{\mathrm{Bub}}(H) = \frac{2\pi\alpha_s^2C_F}{N_cm_c^2}\sum_{m=0}^{\infty}\int_0^{\Lambda}\frac{dx}{2\pi}\,2\,\mathrm{Im}[4m_c^2K(1)]\times 2\,\mathrm{Im}[4m_c^2K^{(m)}(x)]f(x,y)\langle H|\mathcal{O}_1(^1S_0)|H\rangle, \tag{3.19}$$

where $f(x,y)$ is defined in Eq. (2.15), and we have used the approximate relation

$$k_0+\frac{\mathbf{k}^2}{2m_c}\approx k_0-\frac{k^2}{2m_c}, \tag{3.20}$$

which holds for $k_0/m_c\ll 1$. Dividing the rate in Eq. (3.19) by the decay rate into two photons in Eq. (2.13), we obtain the contribution of the color-octet matrix element to R^{Bub}:

$$R_8^{\mathrm{Bub}} = R_0\sum_{m=0}^{\infty}\int_0^{\Lambda}\frac{dx}{2\pi}\frac{dy}{2\pi}\,2\,\mathrm{Im}[4m_c^2K(1)]\times 2\,\mathrm{Im}[4m_c^2K^{(m)}(y)]f(x,y). \tag{3.21}$$

Now we choose the cutoff Λ on the matrix element to be the hard cutoff

$$x\leqslant\delta,$$
$$y\geqslant(1-\sqrt{\delta})^2. \tag{3.22}$$

From Eq. (2.9), we see that this choice of cutoff corresponds to

$$k^2\leqslant 4m_c^2\delta,$$
$$\mathbf{k}^2\leqslant[k^2/(4m_c)+2m_c\sqrt{\delta}-m_c\delta]^2-k^2. \tag{3.23}$$

Then, symmetrizing Eq. (3.21) in x and y, we find that it is precisely the contribution to Eq. (2.14) that is excluded by the restriction (3.9).

We conclude that the contributions that we have set aside by virtue of the restriction (3.9) correspond to contributions to the decay rate that are proportional to the matrix element of the color-octet operator $\mathcal{O}_8(^3S_1)$. In deriving Eq. (3.21), we have dropped contributions of higher order in v. These could be taken into account by including the contributions of higher-dimensional color-octet operators.

We note that the use of a hard cutoff in the color-octet matrix element, as opposed to dimensional regularization, is essential in order to contain the small-virtuality contributions entirely within the matrix element [4]. These contributions have an ultraviolet-divergent power behavior. In dimensional regularization, power-divergent contributions are set to zero, and, therefore, the small-virtuality contributions would be excluded from a dimensionally regulated matrix element.

Using Eqs. (3.14) and (3.15), along with Eqs. (1.3) and (2.13), we see that the contribution of the color-octet operator $\mathcal{O}_8(^3S_1)$ to R is given by

$$R_8 = -2\,\mathrm{Im}[4m_c^2K(1)]R_0\frac{N_c}{\alpha_sC_F}\frac{\langle\eta_c|\mathcal{O}_8(^3S_1)|\eta_c\rangle}{\langle\eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle}. \tag{3.24}$$

We note that, in the limit in which the dimensional regulator ϵ is taken to zero,

$$\mathrm{Im}[4m_c^2K(1)] = \frac{\alpha_s\beta_0\pi}{(1-\alpha_s\beta_0d)^2+\pi^2\alpha_s^2\beta_0^2}. \tag{3.25}$$

In this paper, we will not compute R_8, but, instead, will treat it as an uncertainty in the calculation. The ratio of matrix elements in Eq. (3.24) may be estimated by making use of the velocity-scaling rules for NRQCD [1]. If we adopt the

suggestion of Petrelli *et al.* [14], and include, in addition to the factors of the velocity v, a factor $1/(2N_c)$ in the estimate of the color-singlet matrix element, then we obtain

$$\frac{\langle \eta_c|\mathcal{O}_8(^3S_1)|\eta_c\rangle}{\langle \eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle} \sim \frac{v^3}{2N_c}. \tag{3.26}$$

A one-loop perturbative estimate gives a result that is numerically very close to that in Eq. (3.26):

$$\frac{\langle \eta_c|\mathcal{O}_8(^3S_1)|\eta_c\rangle}{\langle \eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle} \sim \frac{v^3 C_F}{\pi N_c}. \tag{3.27}$$

In making numerical estimates of the size of R_8, we will use the expression (3.27).

The value of the color-octet matrix element depends on the cutoff δ. In making the estimates (3.26) and (3.27), one assumes that the cutoff, in terms of momentum, is of the order of the dynamical scale mv. On the other hand, we would like to choose δ so that nonperturbative contributions arising from momenta less than v are contained in the color-octet operator matrix element, while contributions arising from larger momenta are contained in the short-distance coefficients. We will assume that a momentum cutoff of order mv is consistent with these desiderata. From Eq. (3.23), we see that

$$\mathbf{k}^2/m_c^2 \leqslant (2\sqrt{\delta}-\delta)^2. \tag{3.28}$$

The choice $\delta=0.1$ yields $\mathbf{k}^2/m_c^2 \leqslant 0.28$. For charmonium, 0.28 is very close to the average value of v^2. Therefore, we will use the value $\delta=0.1$ in our calculations.

IV. EFFICIENT METHOD FOR NUMERICAL COMPUTATION

The perturbation series in Eq. (2.28) is convergent. Once we have imposed the restriction (3.9), the series in the first and second terms of Eq. (2.33) are also convergent. However, for reasonable choices of δ, the series converge very slowly. For example, for the second term in Eq. (2.33) with $\delta=0.1$ and $\alpha_s=0.247$, it is necessary to compute through 12th order in α_s in order to achieve an accuracy of about 3%. Furthermore, at high orders in α_s, the integrand becomes strongly oscillatory, and the integral is difficult to compute numerically. It would be far more efficient if we could carry out the perturbation summation before integrating over x and y.

We have already seen in Sec. III A that, in the case of G_1, we can carry out the perturbation summation first, provided that we compensate by adding a contribution that is proportional to the residue at the Landau pole. Using Eqs. (3.1), (3.5) and (3.8), we can write

$$\sum_{n=1}^{\infty} \int_0^1 dx\, g_n(x) = \int_0^1 dx\, g(x) - \frac{z_0}{\alpha_s\beta_0}, \tag{4.1}$$

where

$$g(x) = \sum_{n=1}^{\infty} g_n(x) = \frac{\alpha_s\beta_0}{[1-\alpha_s\beta_0(d-\ln x)]^2+(\alpha_s\beta_0\pi)^2}. \tag{4.2}$$

Then, from Eq. (2.28), we have

$$G_1(\delta) = \frac{1}{\pi\alpha_s\beta_0}\arctan\frac{\pi\alpha_s\beta_0}{1-\alpha_s\beta_0 d} + \int_{(1-\sqrt{\delta})^2}^1 dx\,\frac{g(x)}{x} + \int_0^{(1-\sqrt{\delta})^2} dx\, g(x) - \frac{z_0}{\alpha_s\beta_0}. \tag{4.3}$$

The second term in Eq. (4.3) arises from applying the cutoff (3.9) to G_{1Rb} [Eq. (B3)].

The analysis of Eq. (2.33) is somewhat more involved. For the first term in Eq. (2.33), the restriction on the region of integration (3.9), together with the constraint $\sqrt{x}+\sqrt{y} \leqslant 1$, guarantees that $x,y \geqslant \delta$. Then the sums over n and m are absolutely convergent, provided that δ is sufficiently large that

$$|d-\ln\delta+i\pi|\,\alpha_s\beta_0 < 1. \tag{4.4}$$

For $\alpha_s=0.247$, Eq. (4.4) requires that δ be greater than 0.0482 for the NNA value of d [Eq. (2.31)] and greater than 0.0752 for the background-field-gauge value of d with $\xi=1$ [Eq. (2.32)]. Then, the convergence of the series and the absolute integrability of the sums allow us to interchange the summations and the integrations.

For the second term in Eq. (2.33), we decompose the region of integration according to Eq. (3.12). For the first term on the right side of Eq. (3.12), the perturbation series are absolutely convergent, provided that Eq. (4.4) is satisfied, and we can interchange the summations and the integrations. For the second term on the right side of Eq. (3.12), the x integration lies within the region of absolute convergence of the series, but the y integration does not. However, one can apply the contour-deformation analysis of Sec. III A to the y integral, deforming the contour to a circle of radius δ centered at the origin, so that the entire contour lies within the region of absolute convergence. [The additional factors in the integrand of the second term in Eq. (2.33) contain no singularities within that circle.] It follows that we can interchange the y integration with the summations, provided that we add a compensating term that is proportional to the residue at the Landau pole. Similarly, for the third term on the right side of Eq. (3.12), we can interchange the integrations with the summations, provided that we add a compensating term to the x integral that is proportional to the residue at the Landau pole. For the fourth term on the right side of Eq. (3.12), we must add Landau-pole-residue terms to both the x and y integrals in order to interchange the integrations with the summations. Then, the various terms can be recombined to obtain

$$G_2(\delta) = -\int_0^{(1-\sqrt{\delta})^2} \frac{dx}{x}(1-x)g(x)\int_0^{(1-\sqrt{\delta})^2}\frac{dy}{y} \times(1-y)g(y)\,\theta(\sqrt{x}+\sqrt{y}-1) + \int_0^{(1-\sqrt{\delta})^2}\frac{dx}{x}g(x)\int_0^{(1-\sqrt{\delta})^2}\frac{dy}{y}g(y) \times[f(x,y)-(1-x)(1-y)]\,\theta(1-\sqrt{x}-\sqrt{y}) - \frac{2z_0}{\alpha_s\beta_0} \times\int_0^{(1-\sqrt{\delta})^2}\frac{dx}{x}g(x)[f(x,z_0)-(1-x)(1-z_0)] + \frac{z_0^2}{\alpha_s^2\beta_0^2}[f(z_0,z_0)-(1-z_0)^2]. \quad (4.5)$$

We have verified that the expressions (4.3) and (4.5) agree, within truncation errors, with numerical evaluations of the perturbation sums for G_1 and G_2 through 12th order in α_s.

V. RESULTS

In order to combine our result from the resummation of vacuum-polarization bubbles with the computation through order α_s, we must subtract the part of the contribution through relative order α_s that is contained in the resummation. Expanding G_1 [Eq. (2.28)] through order α_s, we obtain

$$G_1 = 1+\alpha_s\beta_0(1+d)+\ldots. \quad (5.1)$$

The perturbation series for G_2 [Eq. (2.33)] begins at order α_s^2. Therefore, using Eq. (2.27), we see that we must subtract $R_0[1+2\alpha_s\beta_0(1+d)]$. We conclude that the rate R, including the exact next-to-leading-order contribution and bubble resummation of the higher-order contributions, is given by

$$R^{\rm Res} = R_0(\mu)\left\{G_1^2(\mu)+G_2(\mu)+\left(\frac{199}{6}-\frac{13\pi^2}{8}-\frac{8}{9}n_f\right) \times\frac{\alpha_s(\mu)}{\pi}+2\alpha_s(\mu)\beta_0\left[\ln\frac{\mu^2}{4m_c^2}-1-d(\mu)\right]\right\}. \quad (5.2)$$

For the case of NNA resummation, we have

$$R^{\rm NNA} = R_0(\mu)\left\{[G_1^{\rm NNA}(\mu)]^2+G_2^{\rm NNA}(\mu) +\left(\frac{37}{2}-\frac{13\pi^2}{8}\right)\frac{\alpha_s(\mu)}{\pi}\right\}, \quad (5.3)$$

while for the case of background-field-gauge resummation, we have

$$R^{\rm BFG} = R_0(\mu)\left\{[G_1^{\rm BFG}(\mu)]^2+G_2^{\rm BFG}(\mu)+\left[22-\frac{13\pi^2}{8}-\frac{n_f}{3} -\frac{3}{2}(\xi^2-1)\right]\frac{\alpha(\mu)}{\pi}-2\alpha_s(\mu)\beta_0\frac{\alpha_s(\mu)}{\pi}\right\}, \quad (5.4)$$

where ξ is the gauge parameter for fields internal to loops. ($\xi=1$ in the Feynman gauge.)

We now evaluate $R^{\rm Res}$, taking $n_f=3$, choosing $\mu=2m_c$, with m_c the charm-quark pole mass, and using $\alpha_s^{(3)}(2m_c)=0.247\pm0.012$.[2] We use Eqs. (4.3) and (4.5) to compute G_1 and G_2, respectively, with the cutoff $\delta=0.1$.

In the case of the NNA resummation, we obtain

$$G_1^{\rm NNA}=1.67,$$

$$G_2^{\rm NNA}=-0.64,$$

$$R^{\rm NNA}=(3.01\pm0.30\pm0.34)\times10^3, \quad (5.5)$$

where the first uncertainty in $R^{\rm NNA}$ comes from the uncertainty in $\alpha_s(2m_c)$, and the second uncertainty comes from the estimate of the uncalculated color-octet contribution [Eqs. (3.24), (3.25), and (3.27)], with $v=0.3$.

In the case of background-field-gauge resummation, we take the gauge parameter ξ to be equal to unity, since this choice minimizes the size of the residual relative-order-α_s contribution. Then we obtain

$$G_1^{\rm BFG}=1.88,$$

$$G_2^{\rm BFG}=-1.04,$$

$$R^{\rm BFG}=(3.26\pm0.31\pm0.47)\times10^3, \quad (5.6)$$

where the uncertainties in $R^{\rm BFG}$ are as in $R^{\rm NNA}$.

There are additional uncertainties in our results that arise from the dependences on the renormalization scale μ. We expect the scale dependences in the resummed expressions to be considerably less than in the NLO expression. As we have already mentioned, $R_0(\mu)[G_1^2(\mu)+G_2(\mu)]$ is invariant with respect to changes of scale at the level of one-loop running of $\alpha_s(\mu)$. The remaining, unresummed terms in $R^{\rm Res}$ are a small fraction of the total expression and yield a correspondingly small dependence on the scale. If we evolve from $\alpha_s(2m_c)=0.247$ using the one-loop beta function, then we obtain $\alpha_s(m_c)=0.327$. Taking this value of $\alpha_s(m_c)$ and $\mu=m_c$, we obtain a 10.6% increase in the value of $R^{\rm NNA}$ and a 1.8% increase in the value of $R^{\rm BFG}$. At this value of $\alpha_s(m_c)$, the inequality (4.4) is no longer satisfied for the background-field-gauge resummation, and the perturbation expansion no longer converges. For purposes of estimating the scale dependence, we assume that G_1 and G_2 are given by Eqs. (4.3) and (4.5) and, therefore, that $R_0(\mu)[G_1^2(\mu)+G_2(\mu)]$ is scale invariant.

Under three-loop evolution of $\alpha_s(\mu)$, $R_0(\mu)[G_1^2(\mu)+G_2(\mu)]$ is no longer scale invariant, and, hence, the scale

[2]We have obtained this value by evolving with 3-loop accuracy from $\alpha_s(m_Z)=0.118\pm0.002$, where $m_Z=91.188$ GeV is the Z-boson mass, and using for the $\overline{\rm MS}$ charm-quark and bottom-quark masses $m_c^{\overline{\rm MS}}=1.25\pm0.10$ GeV and $m_b^{\overline{\rm MS}}=4.2\pm0.2$ GeV, respectively [7]. The uncertainty in $\alpha_s(2m_c)$ includes the uncertainty in $\alpha_s(M_Z)$ and the uncertainty in the value of m_c.

dependence is larger. Three-loop evolution from $\alpha_s(2m_c) = 0.247$ yields $\alpha_s(m_c) = 0.350$. At this value of $\alpha_s(m_c)$, the inequality (4.4) is not satisfied in either the NNA or background-field-gauge resummations, and we again assume, for purposes of estimating the scale dependence, that G_1 and G_2 are given by Eqs. (4.3) and (4.5). Taking the three-loop-evolved value of $\alpha_s(m_c)$ and $\mu = m_c$, we obtain a 24.5% increase in the value of $R^{\rm NNA}$ and an 11.9% increase in the value of $R^{\rm BFG}$. One could hope to reduce these uncertainties further through a resummation of two-loop bubble contributions. Still, they are already a significant improvement over the factor of 2.4 uncertainty from scale dependence in the NLO result.

VI. SUMMARY AND DISCUSSION

We have studied the ratio R of the hadronic and electronic decay rates for the heavy quarkonium η_c. In R, the dependences of the rates on NRQCD matrix elements cancel at leading order in v. The relative-order v^2 corrections also cancel. Consequently, renormalon ambiguities associated with initial-state virtual gluons cancel at this level of accuracy in v^2 [3,4].

The perturbative corrections to R that are associated with the running of α_s account for most of the large one-loop correction to R. We have carried out a resummation of the leading corrections of this type, which arise from vacuum-polarization insertions in the final-state gluon legs. Our results are in good agreement with the experimental value of R.

There is some arbitrariness in the choice of the resummed quantity. In this paper, we have used the NNA approach [9], in which one resums, in nth order, all contributions proportional to $(\alpha_s\beta_0)^n$. In implementing this method, one resums chains of quark-loop vacuum-polarization bubbles in gluon legs and then accounts for gluon contributions by promoting the quark contribution to β_0 to the full QCD β_0. An alternative to the NNA approach is to resum chains of quark-loop *and* gluon-loop vacuum-polarization bubbles in the background-field gauge [10]. This method is motivated by the fact that, in the background-field gauge, all of the running of α_s is contained in the vacuum-polarization diagrams [12]. The two approaches give numerical results that agree at the level of the other uncertainties in the calculations.

The sum of the vacuum-polarization-bubble terms is independent of the renormalization-scale, at the level of the one-loop running of α_s. Hence, the renormalization-scale dependence of our result is much milder than that of the unresummed one-loop result. However, because of the arbitrariness of the resummation procedure, it would not be correct to conclude that the reduction in the theoretical uncertainty is proportional to the reduction in the scale dependence.

The final-state phase space includes integrations over the virtualities $m_c^2 x$ and $m_c^2 y$ of the final-state gluons. Near zero virtuality, one is in a region in which perturbation theory breaks down and nonperturbative contributions may be important. We have identified this region of low virtuality with contributions to the NRQCD matrix element of a color-octet, ${}^3S_1$ four-fermion operator. The contribution of this matrix element is of relative order v^3, and it is treated as an uncertainty in our calculation.

A key ingredient in our analysis is the demonstration that there are no contributions in which both gluons have small virtualities or in which one gluon is exactly on shell and the other has a small virtuality. Such contributions, if present, would not correspond to an NRQCD operator matrix element and would be inconsistent with the NRQCD factorization program. We demonstrate the absence of such contributions by writing the integrals over the virtualities of the off-shell gluons as contour integrals and deforming the contours out of the regions of small virtuality.

One can understand this contour-deformation argument in terms of the uncertainty principle. In the bubble-chain picture, away from the kinematic limits of maximum gluon virtuality, the quarkonium decay produces two energetic gluon jets, each with energy of order m_c. Consider the case in which both gluons are virtual. That is, each gluon jet contains two subjets that correspond to the constituents of a vacuum-polarization bubble. The mass of each jet ranges up to a value of order m_c. Now suppose that the jets are observed inclusively, that is, without regard to their masses. Provided that neither jet has a mass near the kinematic limit, the virtual-gluon masses are *independently* uncertain by amounts of order m_c and, according to the uncertainty principle, the virtual gluons propagate over distances of order $1/m_c$. Hence, the NRQCD four-fermion annihilation vertex is well localized, in accordance with the NRQCD factorization formula. On the other hand, if one restricts the gluon virtualities (i.e., x and y) to be small, then the gluons propagate over long distances, of order $1/(m_c\sqrt{x})$ and $1/(m_c\sqrt{y})$. Such propagation would delocalize the annihilation vertex. Hence, contributions from small x and y must vanish in the inclusive rate. The absence of singularities in the complex virtuality plane near the vacuum-polarization cut tells us that there are no competing, nearly classical processes that would upset this uncertainty-principle analysis of the bubble chains in isolation.

Note that this uncertainty-principle argument does not rule out contributions in which a gluon is exactly on the mass shell (unless the other gluon is virtual and at small virtuality). One should regard the on-shell gluon not as propagating an infinite distance, but, rather, as following a nearly classical trajectory. Since the momentum and energy are uncertain by amounts of order m_c, the positions and times in the trajectory are smeared by amounts of order $1/m_c$, and one again reaches the conclusion that the NRQCD four-fermion annihilation vertex is localized to within $1/m_c$ [1].

On the other hand, if the mass of one of the jets is near the kinematic limit, then the mass (and uncertainty in mass) of the other jet is constrained kinematically to be small, and the annihilation vertex becomes delocalized. As we have seen, this kinematic region, in which one virtuality is near the kinematic limit, yields long-distance (delocalized) contributions that are absorbed into color-octet operator matrix elements.

An interesting, and somewhat unexpected, feature of the bubble-chain contributions is that their values depend on whether one carries out the integration over the gluon virtu-

alities before or after carrying out the perturbation summation. The two orders of operations lead to results that differ by an amount that has the characteristic structure $\exp[-1/(\alpha_s\beta_0)]$ of a renormalon ambiguity. However, this difference does not arise from a renormalon ambiguity since, in our calculations, all of the renormalon contributions are absorbed into the color-octet, ${}^3S_1$ matrix element. Rather, the difference arises from the failure of the perturbation series to converge in the region of small virtuality. The difference is associated with the presence of the perturbative Landau pole in that region. We have argued, again using contour integration, that one should integrate over the virtualities before carrying out the perturbation summation. Then one can deform contours out of the region of small virtuality and the resulting perturbation series is reliable. On the other hand, there is no justification for summing the perturbation series first: at sufficiently small virtuality, one is outside the radius of convergence, and the series is unreliable. We have argued that the order of operations in which one carries out the integration over the gluon virtualities before carrying out the perturbation summation is equivalent to the standard Borel-transform method for evaluating the contribution of a bubble chain.

For α_s evaluated at the scale $\mu=2m_c$ and for typical values of the cutoff δ, the perturbation series, after integration over the gluon virtualities, converges very slowly. Furthermore, the integrands in high-order terms oscillate strongly and are difficult to evaluate numerically. Therefore, we have devised a more efficient method for numerical computation in which one carries out the perturbation summation before the integrations over the gluon virtualities and then makes corrections to account for the interchange of the order of limits.

Our numerical result for R is much smaller than that which one obtains by applying the BLM scale-setting method to the next-to-leading order calculation.[3] This may seem surprising, since the BLM method is designed to recover the effects associated with the one-loop running of α_s, to the extent that the can be taken into account through a change of renormalization scale. The effects of a change of scale in α_s can be expanded in a perturbation series. At one-loop accuracy in the running of α_s, one obtains the geometric series

$$\alpha_s(\mu') = \frac{\alpha_s(\mu)}{1-2\alpha_s(\mu)\beta_0\ln(\mu/\mu')} = \alpha_s(\mu)\sum_{n=0}^{\infty}[2\alpha_s(\mu)\beta_0\ln(\mu/\mu')]^n. \quad (6.1)$$

It is illuminating to compare this series with the perturbation expansion for the contribution $G_1(\mu)$ [Eq. (2.28)] of a single bubble chain, integrated over the gluon virtuality. In the NNA case we have

$$G_1(2m_c) = 1 + 1.91\alpha_s(2m_c) + 2.47\alpha_s^2(2m_c) + 0.97\alpha_s^3(2m_c) - 4.49\alpha_s^4(2m_c) - 11.76\alpha_s^5(2m_c) + \cdots. \quad (6.2)$$

We see that the individual terms in the expansion (6.2) are *not* well approximated by those of a geometric series. Therefore, we would not expect the BLM method to reproduce this result accurately. Of course, before integration over the gluon virtuality, the bubble chain *is* a geometric series [Eq. (2.7)], with a ratio between terms of $-i\Pi(x)$. However, in order for the bubble-chain contributions, integrated over phase space, to be close to a geometric series, we must have

$$\langle[\Pi(x)]^n\rangle \approx \langle\Pi(x)\rangle^n, \quad (6.3)$$

where the angular brackets denote averaging over phase space, with the other factors in the integrand for R as a weight. The relation (6.3) holds only if the integration over the virtuality x is highly peaked. Hence, we do not expect, in general, to obtain a geometric perturbation series. We note that there are generalizations of the BLM method that take into account higher-order deviations of the perturbation series from a geometric series [15] and that may reproduce the effects computed in this paper.

Finally, we remark that the methods that we have developed in this paper can be applied to a variety of processes to resum large final-state corrections associated with the running of α_s. The methods are most directly applicable to computations in NRQCD, but with small modifications, they are also applicable to computations in full QCD. The principal change in the method that is required in the latter case is to absorb nonperturbative contributions into QCD matrix elements, rather than into NRQCD matrix elements. In general, there are contributions that are associated with the running of α_s in initial-state virtual gluons, as well. Such initial-state contributions also contain nonperturbative pieces that must be absorbed into operator matrix elements.[4] However, the bubble series begins at relative order α_s^2, instead of relative order α_s.

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[3] If one regards the BLM procedure as an approximate resummation of bubble contributions at one-loop accuracy in the bubble, then it would be appropriate to evolve α_s from the nominal scale $2m_c$ to the BLM scale by using the one-loop β function. This leads to $R^{\rm NLO}(\mu_{\rm BLM}) = 6.3\times10^3$, which is smaller than the value, obtained by 3-loop evolution, in Eq. (1.7).

[4] Initial-state contributions associated with the running of α_s have been discussed generally in terms of the Borel-transform method [2] and also for the specific case of NRQCD decays [3,4].

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APPENDIX A: THE GLUON SELF-ENERGY IN THE BACKGROUND-FIELD GAUGE

At one-loop level, a gluon self-energy receives contributions from gluon, ghost and quark bubbles. The value of a gluon bubble depends on the choice of gauge. In this Appendix, we describe the calculation of the gluon self-energy in the background-field gauge [10].

In the background-field-gauge method, the gauge field is separated into an external field or background field, which appears only in external lines of a Feynman diagram, and an internal field, which appears only in loops of a Feynman diagram. The gauge of the internal field is fixed, but that of the external field is not. Hence, the action is explicitly gauge covariant with respect to the external field, and the background-field-gauge method maintains explicit gauge covariance with respect to the external field in each step of a calculation. In the background-field gauge, all of the logarithmic dependence on the renormalization scale that is associated with the running of the coupling constant is contained in the gluon self-energy [12].

In the R_ξ class of Lorentz-covariant background-field gauges, the gauge of the internal fields is specified by the parameter ξ, with $\xi=1$ corresponding to the Feynman gauge [12]. In the background-field R_ξ gauges, the contribution of a gluon bubble to the gluon self-energy is given by

$$\Pi_{\mu\nu}^{ab\ gluon}(k,\xi)=-\frac{1}{2}g_s^2\mu^{2\epsilon}f^{acd}f^{bdc}\int\frac{d^Dp}{(2\pi)^D}\frac{N_{\mu\nu}(p,k,\xi)}{p^2(p+k)^2}, \tag{A1}$$

where

$$\begin{aligned}N_{\mu\nu}(p,k,\xi)=&\Big\{(2p+k)_\mu g_{\alpha\beta}+2k_\alpha g_{\mu\beta}-2k_\beta g_{\mu\alpha}\\&+\frac{1-\xi}{\xi}[(k+p)_\alpha g_{\mu\beta}+p_\beta g_{\mu\alpha}]\Big\}\\&\times\Big\{(2p+k)_\nu g_{\gamma\delta}+2k_\gamma g_{\nu\delta}-2k_\delta g_{\nu\gamma}\\&+\frac{1-\xi}{\xi}[(k+p)_\gamma g_{\nu\delta}+p_\delta g_{\nu\gamma}]\Big\}\\&\times\left[g^{\alpha\gamma}-(1-\xi)\frac{(p+k)^\alpha(p+k)^\gamma}{(p+k)^2}\right]\\&\times\left[g^{\beta\delta}-(1-\xi)\frac{p^\beta p^\delta}{p^2}\right].\end{aligned} \tag{A2}$$

A straightforward calculation, using Feynman parameter integrals, gives

$$\begin{aligned}\Pi_{\mu\nu}^{ab\ gluon}(k,\xi)=&-i(k^2g_{\mu\nu}-k_\mu k_\nu)\delta^{ab}\frac{\alpha_s}{2\pi}C_A\left(\frac{4\pi\mu^2}{-k^2-i\epsilon}\right)^\epsilon\\&\times\Big\{\Gamma(\epsilon)\Gamma(1-\epsilon)\Big[\frac{\Gamma(3-\epsilon)}{\Gamma(4-2\epsilon)}\\&-2\,\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\Big]+\frac{1}{2}(\xi^2-1)\Big\}.\end{aligned} \tag{A3}$$

The contribution of a ghost bubble to the gluon self-energy in the background-field gauge is

$$\begin{aligned}\Pi_{\mu\nu}^{ab\ ghost}(k)=&\,g_s^2\mu^{2\epsilon}f^{acd}f^{bdc}\int\frac{d^Dp}{(2\pi)^D}\frac{(2p+k)_\mu(2p+k)_\nu}{p^2(p+k)^2}\\=&\,i(k^2g_{\mu\nu}-k_\mu k_\nu)\delta^{ab}\frac{\alpha_s}{2\pi}C_A\left(\frac{4\pi\mu^2}{-k^2-i\epsilon}\right)^\epsilon\\&\times\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(4-2\epsilon)}.\end{aligned} \tag{A4}$$

The contribution of n_f massless quarks to the gluon self-energy is

$$\begin{aligned}\Pi_{\mu\nu}^{ab\ quark}(k)=&-g_s^2n_f\mu^{2\epsilon}\mathrm{Tr}(T^aT^b)\\&\times\int\frac{d^Dp}{(2\pi)^D}\frac{\mathrm{Tr}[(\not p+\not k)\gamma_\mu\not p\gamma_\nu]}{p^2(p+k)^2}\\=&-in_f(k^2g_{\mu\nu}-k_\mu k_\nu)\delta^{ab}\frac{\alpha_s}{\pi}\left(\frac{4\pi\mu^2}{-k^2-i\epsilon}\right)^\epsilon\\&\times\frac{\Gamma(\epsilon)\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)}.\end{aligned} \tag{A5}$$

Adding Eqs. (A3), (A4) and (A5) and subtracting the pole in ϵ that corresponds to an ultraviolet divergence, along with an associated constant that depends on the renormalization scheme, we obtain the expression for the renormalized one-loop gluon self-energy:

$$\Pi_{\mu\nu}^{ab}(k,\xi)=i(k^2g_{\mu\nu}-k_\mu k_\nu)\delta^{ab}\beta_0\alpha_s\frac{1}{\epsilon}(bx^{-\epsilon}-a), \tag{A6}$$

where

$$\begin{aligned}b=&\frac{\Gamma(1+\epsilon)}{\beta_0\pi}\Big[C_A\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}-\left(\frac{C_A}{2}+n_f\right)\frac{\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)}\\&-\frac{C_A\epsilon}{4\beta_0\pi}(\xi^2-1)\Big]\left(\frac{\pi\mu^2}{-m_c^2-i\epsilon}\right)^\epsilon\\=&\left\{1+\left[-\gamma+\ln 4\pi+\frac{1}{\beta_0\pi}\left(\frac{67}{12}-\frac{5}{18}n_f-\frac{3}{4}(\xi^2-1)\right)\right.\right.\\&\left.\left.+\ln\frac{\mu^2}{4m_c^2}\right]\epsilon+O(\epsilon^2)\right\}e^{i\pi\epsilon},\end{aligned} \tag{A7a}$$

$$a=1+\left(-\gamma+\ln 4\pi+\frac{5}{3}+C\right)\epsilon+O(\epsilon^2), \tag{A7b}$$

$\beta_0=(33-2n_f)/(12\pi)$, $x\equiv p^2/(4m_c^2)$, and C is a renormalization-scheme-dependent constant. In the $\overline{\rm MS}$ scheme, $C=-5/3$. Now, in Eqs. (A6) and (A7), ϵ plays the role of an infrared regulator ($\epsilon<0$).

APPENDIX B: COMPUTATION OF A SINGLE BUBBLE CHAIN

In this Appendix, we compute the real and virtual contributions from a single bubble chain. We consider first the real contribution. From Eq. (2.23), we have

$$G_{1R}=\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^1 dx\left(1-\frac{1}{x}\right)\mathrm{Im}\left[\frac{1}{\epsilon}\alpha_s(bx^{-\epsilon}-a)\right]^n, \tag{B1}$$

where $\epsilon<0$ is an infrared regulator. The two terms in parentheses have different infrared behaviors. The contribution of the second term contains infrared divergences, which appear as poles in ϵ. The contribution of the first term is free of infrared divergences, and so we can take $\epsilon=0$ in the integrand. Thus, the contribution of the first term can be written as

$$G_{1Ra}=\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^1 dx\,\mathrm{Im}[\alpha_s\beta_0(d-\ln x+i\pi)]^n, \tag{B2}$$

where d is defined in Eq. (2.30).

Now let us evaluate the second term in parentheses in Eq. (B1). It can be written as

$$\begin{aligned}G_{1Rb}&=-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\alpha_s\beta_0)^n}{\epsilon^n}\,\mathrm{Im}\int_0^1\frac{dx}{x}(bx^{-\epsilon}-a)^n\\&=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\alpha_s\beta_0)^n}{\epsilon^{n+1}}\,\mathrm{Im}\int_0^1 dy\,b\frac{[(by-a)^n-(-a)^n]}{(by-a)-(-a)}\\&=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\alpha_s\beta_0)^n}{\epsilon^{n+1}}\,\mathrm{Im}\int_0^1 dy\,b\\&\quad\times\sum_{m=0}^{n-1}(-a)^{(n-m-1)}(by-a)^m,\end{aligned} \tag{B3}$$

where we have made a change of integration variable $y=x^{-\epsilon}$, and, in the second line, we have used the fact that a is real. Carrying out the integration over y, we obtain

$$\begin{aligned}G_{1Rb}&=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\alpha_s\beta_0)^n}{\epsilon^{n+1}}\,\mathrm{Im}\sum_{m=1}^{n}\frac{1}{m}(-a)^{(n-m)}(b-a)^m\\&=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-\alpha_s\beta_0 a)^n}{\epsilon^{n+1}}\,\mathrm{Im}\left[-\ln\frac{b}{a}-\frac{1}{n+1}\left(\frac{a-b}{a}\right)^{n+1}+O(\epsilon^{n+2})\right]\\&=-\sum_{n=1}^{\infty}\frac{(-\alpha_s\beta_0 a)^n}{\epsilon^n}\\&\quad+\frac{1}{\pi\alpha_s\beta_0}\sum_{n=2}^{\infty}\frac{1}{n}\,\mathrm{Im}\left(\alpha_s\beta_0\frac{b-a}{\epsilon}\right)^n+O(\epsilon),\end{aligned} \tag{B4}$$

where, in the last line, we have used the fact that the phase of b, exact to all orders in ϵ, is $\pi\epsilon$. In the last line of Eq. (B4), the first term contains infrared divergences, and, since $b-a\sim\epsilon$, the second term is infrared finite.

From Eq. (2.24), we see that the virtual correction from a single bubble chain is

$$G_V=\sum_{n=1}^{\infty}\frac{(-\alpha_s\beta_0 a)^n}{\epsilon^n}. \tag{B5}$$

The terms in Eq. (B5) with $n\geqslant 1$ exactly cancel the the infrared divergences in the first term of the last line of Eq. (B4). The $n=0$ term of Eq. (B5) and the remainder of Eq. (B4) combine to give

$$\begin{aligned}G_{1Rb}+G_V&=\frac{-1}{\pi\alpha_s\beta_0}\,\mathrm{Im}\ln\left[1-\alpha_s\beta_0\frac{b-a}{\epsilon}\right]\\&=\frac{1}{\pi\alpha_s\beta_0}\arctan\frac{\pi\alpha_s\beta_0}{1-\alpha_s\beta_0 d}.\end{aligned} \tag{B6}$$

Combining Eqs. (B3) and (B6), we obtain

$$\begin{aligned}G_1&=G_{1Ra}+G_{1Rb}+G_V\\&=\frac{1}{\pi\alpha_s\beta_0}\arctan\frac{\pi\alpha_s\beta_0}{1-\alpha_s\beta_0 d}\\&\quad+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^1 dx\,\mathrm{Im}[\alpha_s\beta_0(d-\ln x+i\pi)]^n.\end{aligned} \tag{B7}$$

APPENDIX C: RENORMALON IN THE SHORT-DISTANCE COEFFICIENTS

In this Appendix, we compute the leading renormalon singularities in the Borel transforms of the short-distance coefficients of R.[5] We consider the expressions for G_1 and G_2, Eqs. (2.28) and (2.33), without the cutoff (3.9). Once the

[5]For reviews of the properties of renormalons and Borel transforms, see Refs. [2,4].

cutoff (3.9) has been imposed, the renormalon singularities are absent in G_1 and G_2.

The Borel transform of a function $f(\alpha_s)$ that has a perturbation expansion

$$f(\alpha_s)=\sum_{n=0}^{\infty} a_n \alpha_s^n \tag{C1}$$

is defined by

$$B[f](u)=a_0\delta(u/\beta_0)+\sum_{n=1}^{\infty}\frac{a_n}{(n-1)!}(u/\beta_0)^{n-1}. \tag{C2}$$

The renormalon singularities in the Borel transform are those that arise from the bubble-chain sum, and the leading singularity is the one that appears at the smallest value of u.

Renormalon singularities can arise from the ultraviolet or infrared regions of momentum integrals. Since, in R, the virtualities x and y are bounded by unity, there are no ultraviolet renormalons. We have argued (in Sec. III A) that G_1 receives no contributions from the region of small virtuality. Hence, it contains no renormalons. We have also argued (in Sec. III B) that G_2 receives no contributions in which both x and y are small. Therefore, the renormalons in G_2, can arise only from the regions $x\to 0$ and $y\sim 1$ or $y\to 0$ and $x\sim 1$.

We consider the region $x\to 0$ and $y\sim 1$ and multiply our result by 2 to account for the contribution of the second region. In this region, in the expression (2.33), we can approximate $g_n(y)$ by $g_n(1)$. Corrections to this yield behavior in x that is subleading as $x\to 0$. Furthermore, the $g_n(1)$ yield a convergent series, and, so, we may carry out the Borel transform term by term. From the relation

$$B[\alpha_s f](u)=\frac{1}{\beta_0}\int_0^u du' B[f](u'), \tag{C3}$$

we see that the leading singularity in the Borel transform of G_2 must come from the leading power of α_s in the terms $g_n(1)$: the integration on the right side of Eq. (C3) weakens the singularity as the power of α_s from the $g_n(1)$ increases. Therefore, we retain only the leading term $g_1(1)=\alpha_s\beta_0$, to obtain

$$\begin{aligned}\frac{1}{2}B[G_2](u)\sim&-\beta_0\int_0^1\frac{dx}{x}(1-x)B[\alpha_s g](x,u)\\&\times\int_{(1-\sqrt{x})^2}^1\frac{dy}{y}(1-y)+\beta_0\int_0^1\frac{dx}{x}B[\alpha_s g](x,u)\\&\times\int_0^{(1-\sqrt{x})^2}\frac{dy}{y}[f(x,y)-(1-x)(1-y)],\end{aligned} \tag{C4}$$

where

$$\begin{aligned}B[\alpha_s g](x,u)&=\frac{1}{\pi}\,\mathrm{Im}(x^{-u}e^{du+i\pi u})\\&=x^{-u}e^{du}\sin(\pi u).\end{aligned} \tag{C5}$$

An analysis of the integrations over y yields the leading contributions for x small. Inserting these results into Eq. (C4), we have

$$\begin{aligned}\frac{1}{2}B[G_2](u)\sim&-\beta_0\int_0^1\frac{dx}{x}(1-x)B[\alpha_s g](x,u)(2/3)x^{3/2}\\&+\beta_0\int_0^1\frac{dx}{x}B[\alpha_s g](x,u)3x\ln x,\end{aligned} \tag{C6}$$

where the first and second terms on the right side of Eq. (C6) correspond to the first and second terms, respectively, on the right side of Eq. (C4). Then, a straightforward calculation gives the leading renormalon singularity for each of the two terms in Eq. (C4):

$$B[G_2](u)\sim\frac{\beta_0}{\pi}e^{du}\left[-\frac{8}{3}\frac{\sin(\pi u)}{3-2u}-6\frac{\sin(\pi u)}{(1-u)^2}\right], \tag{C7}$$

where the first and second terms on the right side of Eq. (C7) correspond to the first and second terms, respectively, on the right sides of Eqs. (C4) and (2.33).

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